

Coxeter groups, symmetries, and rooted representations

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1 Introduction

Let Γ be a Coxeter graph, and let (W, S) be the Coxeter system of Γ . A *symmetry* of Γ is a permutation g of S such that $m_{g(s),g(t)} = m_{s,t}$ for all $s, t \in S$, where $(m_{s,t})_{s,t \in S}$ is the Coxeter matrix of Γ . Let G be a group of symmetries of Γ . Then G is necessarily finite, and it can be viewed as a group of automorphisms of W . We denote by W^G the subgroup of W fixed under the action of G . Mühlherr [7] and Hée [4], independently of one another, proved that W^G is a Coxeter group. None of them gave explicitly the Coxeter graph $\tilde{\Gamma}$ which defines W^G . However, a third proof, different from the other two, with an explicit description of $\tilde{\Gamma}$, is given in Crisp [2, 3].

Let $\Pi = \{\epsilon_s \mid s \in S\}$ be a set in one-to-one correspondence with S , let V be the real vector space having Π as a basis, and let $\langle \cdot, \cdot \rangle$ be the symmetric bilinear form on V defined by $\langle \epsilon_s, \epsilon_t \rangle = -\cos(\pi/m_{s,t})$ if $m_{s,t} \neq \infty$ and $\langle \epsilon_s, \epsilon_t \rangle = -1$ if $m_{s,t} = \infty$. For every $s \in S$ we define the linear transformation $f_s : V \rightarrow V$ by $f_s(x) = x - 2\langle x, \epsilon_s \rangle \epsilon_s$. Then the map $S \rightarrow \text{GL}(V)$, $s \mapsto f_s$, induces a celebrated faithful linear representation $f : W \rightarrow \text{GL}(V)$, called the *canonical representation* of W (see Bourbaki [1]). In our context, the triple $(V, \langle \cdot, \cdot \rangle, \Pi)$ will be called the *canonical root basis* of Γ .

Let G be a group of symmetries of Γ . Then G acts on V sending ϵ_s to $\epsilon_{g(s)}$ for all $s \in S$, and this action leaves invariant the canonical form $\langle \cdot, \cdot \rangle$. Hence, the canonical representation $f : W \rightarrow \text{GL}(V)$ is equivariant, in the sense that $f(g(w)) = g \circ f(w) \circ g^{-1}$ for all $g \in G$ and all $w \in W$, and therefore f induces a linear representation $f^G : W^G \rightarrow \text{GL}(V^G)$, where $V^G = \{x \in V \mid g(x) = x \text{ for all } g \in G\}$.

A naive question would be: Is $f^G : W^G \rightarrow \text{GL}(V^G)$ the canonical representation of W^G ? A positive answer would provide a way to (re)prove that W^G is a Coxeter group and to determine the Coxeter graph of W^G . Unfortunately, simple calculations show that f^G is not the canonical representation in general. Nevertheless, one can transpose

this question to a larger family of linear representations, the rooted representations introduced by Krammer [5, 6], and, in this context, the answer is yes. Our purpose is to show that.

A *root basis* of Γ is a triple $(V, \langle \cdot, \cdot \rangle, \Pi)$, where V is a finite dimensional real vector space, $\langle \cdot, \cdot \rangle$ is a symmetric bilinear form on V , and $\Pi = \{\epsilon_s \mid s \in S\}$ is a collection of vectors in V in one-to-one correspondence with S , that satisfies the following properties:

- (a) $\langle \epsilon_s, \epsilon_s \rangle = 1$ for all $s \in S$;
- (b) for all $s, t \in S$, $s \neq t$, we have

$$\begin{aligned} \langle \epsilon_s, \epsilon_t \rangle &= -\cos(\pi/m_{s,t}) & \text{if } m_{s,t} \neq \infty, \\ \langle \epsilon_s, \epsilon_t \rangle &\in (-\infty, -1] & \text{if } m_{s,t} = \infty; \end{aligned}$$
- (c) there exists $\chi \in V^*$ such that $\chi(\epsilon_s) > 0$ for all $s \in S$.

As mentioned above, if Π is a basis of V and $\langle \epsilon_s, \epsilon_t \rangle = -1$ whenever $m_{s,t} = \infty$, then $(V, \langle \cdot, \cdot \rangle, \Pi)$ is called the *canonical root basis* of Γ .

This definition is taken from Krammer's thesis [5, 6]. It is both, a generalization of the canonical spaces and canonical forms defined by Bourbaki [1], and a new point of view on the theory of reflection groups developed by Vinberg [8]. Note also that Condition (c) in the above definition often follows from Conditions (a) and (b), but not always (see Krammer [6, Proposition 6.1.2]).

Let $(V, \langle \cdot, \cdot \rangle, \Pi)$ be a root basis of Γ . For every $s \in S$ we define the linear transformation $f_s : V \rightarrow V$ by $f_s(x) = x - 2\langle x, \epsilon_s \rangle \epsilon_s$. The following theorem can be proved for any root basis in the same way as it is proved in Bourbaki [1] for the canonical root basis.

Theorem 1.1 (Krammer [5, 6]) *The map $S \rightarrow \text{GL}(V)$, $s \mapsto f_s$, induces a faithful linear representation $f : W \rightarrow \text{GL}(V)$.*

The representation $f : W \rightarrow \text{GL}(V)$ of Theorem 1.1 is called the *rooted representation* of W associated with $(V, \langle \cdot, \cdot \rangle, \Pi)$.

Let G be a group of symmetries of Γ , and let $(V, \langle \cdot, \cdot \rangle, \Pi)$ be a root basis. As for the canonical root basis, we assume that G embeds in $\text{GL}(V)$, satisfies $g(\epsilon_s) = \epsilon_{g(s)}$ for all $g \in G$ and all $s \in S$, and leaves invariant the form $\langle \cdot, \cdot \rangle$. Then the representation $f : W \rightarrow \text{GL}(V)$ is equivariant in the sense that $f(g(w)) = g \circ f(w) \circ g^{-1}$ for all $g \in G$ and all $w \in W$, and therefore f induces a linear representation $f^G : W^G \rightarrow \text{GL}(V^G)$, where $V^G = \{x \in V \mid g(x) = x \text{ for all } g \in G\}$. The goal of this paper is to prove the following.

Theorem 1.2 (1) *The group W^G is a Coxeter group.*

(2) *Let $\tilde{\Gamma}$ denote the Coxeter graph of W^G . There exists a subset $\tilde{\Pi}$ of V^G such that $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$ is a root basis of $\tilde{\Gamma}$, and the induced representation $f^G : W^G \rightarrow \text{GL}(V^G)$ is the rooted representation associated with $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$. In particular, f^G is faithful.*

A similar approach is adopted in Hée [4, Section 3], with different definitions. One can easily show that the root system obtained from a root basis is a root system in Hée sense [4], and that part of the results of the paper, such as the fact that $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$ is a root basis, can be deduced from Hée [4]. However, to get the explicit expression of the Coxeter graph $\tilde{\Gamma}$ of W^G , one would need extra arguments that can be either a rewrite of Lemma 3.3, or some arguments similar to that given in Crisp [2, 3]. More generally, the whole theorem is more or less in the literature. In particular, as mentioned before, Part (1) is explicit in Mühlherr [7], Hée [4] and Crisp [2, 3]. But, our aim is to provide a new point of view on the question with unified, short and self-contained proofs.

A more precise statement of Theorem 1.2 is given in Section 2. In particular, the Coxeter graph $\tilde{\Gamma}$ and the set $\tilde{\Pi}$ are explicitly described. Section 3 is dedicated to the proofs.

2 Statement

The *length* of an element $w \in W$, denoted by $\text{lg}(w)$, is the shortest length of an expression of w over the elements of S . An expression $w = s_1 \cdots s_\ell$ is called *reduced* if $\ell = \text{lg}(w)$. It is known that, if W is finite, then W has a unique *longest element*, that is, an element $w_0 \in W$ such that $\text{lg}(w) \leq \text{lg}(w_0)$ for all $w \in W$, and this element is an involution (see Bourbaki [1]).

For $X \subset S$, we denote by Γ_X the full subgraph of Γ spanned by X , and by W_X the subgroup of W generated by X . The subgroup W_X is called a *standard parabolic subgroup* of W . By Bourbaki [1], (W_X, X) is a Coxeter system of Γ_X . If W_X is finite, then we denote by w_X the longest element of W_X .

Let G be a group of symmetries of Γ . Now, we define a Coxeter matrix $\tilde{M} = \tilde{M}^G = (\tilde{m}_{X,Y})_{X,Y \in \mathcal{S}}$ (and its associated Coxeter graph, $\tilde{\Gamma}$). This will be the Coxeter matrix (and the Coxeter graph) of W^G (see Theorem 1.2 and Theorem 2.2).

We denote by \mathcal{O} the set of orbits of G in S , and we set $\mathcal{S} = \{X \in \mathcal{O} \mid W_X \text{ is finite}\}$. Then \mathcal{S} is the set of indices of \tilde{M} (which is the set of vertices of $\tilde{\Gamma}$). Let $X, Y \in \mathcal{S}$.

- If $m_{s,t} = 2$ for all $s \in X$ and all $t \in Y$, then we set $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = 2$.
- Let $k \in \{1, 2, 3, 4, 5\}$. If $\Gamma_{X \cup Y}$ is a disjoint union of copies of the Coxeter graph depicted in Figure 2.1 (k), where the vertices corresponding to x_1, x_2, \dots belong to X and the vertices corresponding to y_1, y_2, \dots belong to Y , then we set $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = m$ if $k = 1$, $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = 4$ if $k \in \{2, 3\}$, $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = 8$ if $k = 4$, and $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = 6$ if $k = 5$. In this case we say that (X, Y) is a *bi-orbit of type k* .
- We set $\tilde{m}_{X,Y} = \tilde{m}_{Y,X} = \infty$ in the remaining cases.

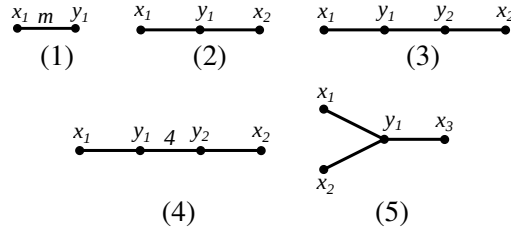


Figure 2.1: Bi-orbits.

The next lemma will be used in the definition of the set $\tilde{\Pi}$. It is well-known and can be easily proved using [1, Chapter V, Section 4, Subsection 8].

Lemma 2.1 *Let $(V, \langle \cdot, \cdot \rangle, \Pi)$ be a root basis of Γ . Suppose that W is finite and that Π spans V . Then $\langle \cdot, \cdot \rangle$ is a scalar product, and $(V, \langle \cdot, \cdot \rangle, \Pi)$ is the canonical root basis of Γ . In particular, Π is a basis of V .*

We turn back to the hypothesis of Theorem 1.2, that is, Γ is any Coxeter graph, G is a group of symmetries of Γ , and $(V, \langle \cdot, \cdot \rangle, \Pi)$ is a root basis of Γ . We assume that G embeds in $\text{GL}(V)$ so that the form $\langle \cdot, \cdot \rangle$ is invariant under the action of G , and $g(\epsilon_s) = \epsilon_{g(s)}$ for all $s \in S$ and all $g \in G$.

Let X be an element of \mathcal{S} , that is, an orbit of G in S such that W_X is finite. Set $\Pi_X = \{\epsilon_s \mid s \in X\}$, and denote by V_X the linear subspace of V spanned by Π_X , and by $\langle \cdot, \cdot \rangle_X$ the restriction of $\langle \cdot, \cdot \rangle$ to $V_X \times V_X$. By Lemma 2.1, Π_X is a basis of V_X and $\langle \cdot, \cdot \rangle_X$ is a scalar product. Let $a_X = \sum_{s \in X} \epsilon_s$. Note that $a_X \in V^G$, hence, by the above, $a_X \neq 0$ and $\|a_X\| > 0$. We set $\tilde{\epsilon}_X = \frac{a_X}{\|a_X\|}$ for all $X \in \mathcal{S}$, and $\tilde{\Pi} = \tilde{\Pi}^G = \{\tilde{\epsilon}_X \mid X \in \mathcal{S}\}$. The main result of the paper, with a precise statement, is the following.

Theorem 2.2 (1) *The set $\mathcal{S}_W = \{w_X \mid X \in \mathcal{S}\}$ generates W^G , and (W^G, \mathcal{S}_W) is a Coxeter system of $\tilde{\Gamma}$.*

- (2) The triple $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$ is a root basis of $\tilde{\Gamma}$, and the induced representation $f^G : W^G \rightarrow \text{GL}(V^G)$ is the rooted representation associated with $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$. In particular, f^G is faithful.

Remark The proof of Part (1) of Theorem 2.2 uses the induced representation $f^G : W^G \rightarrow \text{GL}(V^G)$. Nevertheless, the conclusion of Part (1) is always true because there is always a root basis which satisfies the hypothesis of the theorem: the canonical root basis.

3 Proof

We assume given a Coxeter graph Γ , a root basis $(V, \langle \cdot, \cdot \rangle, \Pi)$ of Γ , and a group G of symmetries of Γ . We assume that G embeds in $\text{GL}(V)$, satisfies $g(\epsilon_s) = \epsilon_{g(s)}$ for all $g \in G$ and all $s \in S$, and leaves invariant the form $\langle \cdot, \cdot \rangle$.

Let $f : W \rightarrow \text{GL}(V)$ be the rooted representation of W associated with $(V, \langle \cdot, \cdot \rangle, \Pi)$. From now on, in order to simplify the notations, we will assume that W acts on V via f , and we will write $w(x)$ in place of $f(w)(x)$ for $w \in W$ and $x \in V$. Lemmas 3.1 to 3.4 are preliminaries to the proof of Theorem 2.2. Lemma 3.1 (1) is well-known. It is a direct consequence of Mühlherr [7, Lemma 2.8], and its proof can be found in the beginning of the proof of Mühlherr [7, Theorem 1.3]. Lemma 3.1 (2) is also known. Its proof is implicit in Crisp [2], but, as far as we know, it is not explicitly given anywhere else.

Lemma 3.1 (1) The group W^G is generated by S_W .

- (2) We have $(w_X w_Y)^{\tilde{m}_{X,Y}} = 1$ for all $X, Y \in \mathcal{S}$ such that $\tilde{m}_{X,Y} \neq \infty$.

Proof As mentioned above, the proof of Part (1) can be found in Mühlherr [7]. So, we only need to prove Part (2). Let $X \subset S$ be such that Γ_X is a disjoint union of vertices (i.e. Γ_X has no edge). Then W_X is finite and $w_X = \prod_{s \in X} s$. Let $X = \{s, t\}$ be a pair included in S such that $m_{s,t} = m < \infty$. Then W_X is finite, $w_X = (st)^{\frac{m}{2}}$ if m is even, and $w_X = (st)^{\frac{m-1}{2}} s$ if m is odd. Now, let $X, Y \in \mathcal{S}$. If $m_{s,t} = 2$ for all $s \in X$ and $t \in Y$, then w_X and w_Y commute, hence $(w_X w_Y)^2 = 1$, as w_X and w_Y are both involutions. Suppose that (X, Y) is a bi-orbit of type j , where $j \in \{1, 2, 3, 4, 5\}$. Let $\Gamma_1, \dots, \Gamma_\ell$ be the connected components of $\Gamma_{X \cup Y}$. For $i \in \{1, \dots, \ell\}$, we denote by Z_i the set of vertices of Γ_i , and we set $X_i = X \cap Z_i$ and $Y_i = Y \cap Z_i$. We have $w_X = \prod_{i=1}^\ell w_{X_i}$ and $w_Y = \prod_{i=1}^\ell w_{Y_i}$. Moreover, using the above observation together

with Theorem 1.1, it is easily checked that $(w_{X_i} w_{Y_i})^{\tilde{m}_{X,Y}} = 1$ for all i . It follows that $(w_X w_Y)^{\tilde{m}_{X,Y}} = \prod_{i=1}^{\ell} (w_{X_i} w_{Y_i})^{\tilde{m}_{X,Y}} = 1$. \square

Lemma 3.2 *Let $X \in \mathcal{S}$. Then one of the following two alternatives holds.*

- (I) Γ_X is a disjoint union of vertices (i.e. Γ_X has no edge).
- (II) There exists $m \in \mathbb{N}$, $m \geq 3$, such that Γ_X is a disjoint union of copies of the Coxeter graph depicted in Figure 2.1 (I).

Proof For $s \in X$ we set $v_s(X) = |\{t \in X \mid m_{s,t} \geq 3\}|$. Since W_X is finite, the connected components of Γ_X are trees (see Bourbaki [1]), hence there exists $s \in X$ such that $v_s(X) \leq 1$. On the other hand, since G acts transitively on X , we have $v_s(X) = v_t(X)$ for all $s, t \in X$. So, either $v_s(X) = 0$ for all $s \in X$, or $v_s(X) = 1$ for all $s \in X$. If $v_s(X) = 0$ for all $s \in X$, then we are in Alternative (I). If $v_s(X) = 1$ for all $s \in X$, then we are in Alternative (II). \square

Let $X \in \mathcal{S}$. We say that X is of *type I* if Γ_X satisfies Condition (I) of Lemma 3.2, and that X is of *type II_m* if Γ_X satisfies Condition (II).

Lemma 3.3 *Let $X, Y \in \mathcal{S}$, $X \neq Y$. Then*

$$\begin{aligned} \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle &= -\cos(\pi/\tilde{m}_{X,Y}) && \text{if } \tilde{m}_{X,Y} \neq \infty, \\ \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle &\in (-\infty, -1] && \text{if } \tilde{m}_{X,Y} = \infty. \end{aligned}$$

Proof Observe that, if $m_{s,t} = 2$ for all $s \in X$ and all $t \in Y$, then $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = 0$ and $\tilde{m}_{X,Y} = 2$. Hence, we can assume that there exist $s \in X$ and $t \in Y$ such that $m_{s,t} \geq 3$. Since G acts transitively on X and leaves invariant Y , it follows that, for all $s \in X$, there exists $t \in Y$ such that $m_{s,t} \geq 3$. Similarly, for all $t \in Y$, there exists $s \in X$ such that $m_{s,t} \geq 3$.

Recall that $a_X = \sum_{s \in X} \epsilon_s$, $a_Y = \sum_{t \in Y} \epsilon_t$, $\tilde{\epsilon}_X = \frac{a_X}{\|a_X\|}$, $\tilde{\epsilon}_Y = \frac{a_Y}{\|a_Y\|}$. Choose $s \in X$ and set $v_X = |\{t \in Y \mid m_{s,t} \geq 3\}|$ and $p_X = \sum_{t \in Y} \langle \epsilon_s, \epsilon_t \rangle = \langle \epsilon_s, a_Y \rangle$. Since G acts transitively on X and leaves invariant Y , these definitions do not depend on the choice of s . Similarly, choose $t \in Y$ and set $v_Y = |\{s \in X \mid m_{s,t} \geq 3\}|$ and $p_Y = \sum_{s \in X} \langle \epsilon_t, \epsilon_s \rangle = \langle \epsilon_t, a_X \rangle$. The hypothesis that there exist $s \in X$ and $t \in Y$ such that $m_{s,t} \geq 3$ implies that $v_X \geq 1$ and $v_Y \geq 1$.

Let $s \in X$ and $t \in Y$. If $m_{s,t} \geq 3$, then $\langle \epsilon_s, \epsilon_t \rangle \leq -\frac{1}{2}$, and if $m_{s,t} = 2$, then $\langle \epsilon_s, \epsilon_t \rangle = 0$. It follows that

$$(3-1) \quad p_X \leq -\frac{v_X}{2}.$$

On the other hand, we have

$$(3-2) \quad |X| v_X = |Y| v_Y.$$

This is the number of edges in Γ connecting an element of X with an element of Y . A direct calculation shows that

$$(3-3) \quad \|a_X\| = \begin{cases} \sqrt{|X|} & \text{if } X \text{ is of type } I, \\ \sqrt{|X|(1 - \cos(\pi/m))} & \text{if } X \text{ is of type } II_m. \end{cases}$$

Finally, by definition of p_X ,

$$(3-4) \quad \langle a_X, a_Y \rangle = |X| p_X.$$

Case 1: X and Y are of type I . Applying Equations (3-2), (3-3), and (3-4) we get

$$(3-5) \quad \langle \tilde{e}_X, \tilde{e}_Y \rangle = \frac{p_X \sqrt{v_Y}}{\sqrt{v_X}}.$$

Applying Equation (3-1) to this equality we get $\langle \tilde{e}_X, \tilde{e}_Y \rangle \leq -\frac{\sqrt{v_X v_Y}}{2}$. It follows that, if either $v_X \geq 4$, or $v_Y \geq 4$, or $v_X, v_Y \geq 2$, then $\langle \tilde{e}_X, \tilde{e}_Y \rangle \leq -1$. If $v_X = 1$, $v_Y \geq 2$ and $p_X \leq -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$, then, by Equation (3-5), $\langle \tilde{e}_X, \tilde{e}_Y \rangle \leq -1$. If $v_X = 1$, $v_Y = 3$ and $p_X = -\cos(\pi/3) = -\frac{1}{2}$, then, by Equation (3-5), $\langle \tilde{e}_X, \tilde{e}_Y \rangle = -\frac{\sqrt{3}}{2} = -\cos(\pi/6)$. In this case (Y, X) is a bi-orbit of type 5 and $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = 6$. If $v_X = 1$, $v_Y = 2$ and $p_X = -\cos(\pi/3) = -\frac{1}{2}$, then, by Equation (3-5), $\langle \tilde{e}_X, \tilde{e}_Y \rangle = -\frac{\sqrt{2}}{2} = -\cos(\pi/4)$. In this case (Y, X) is a bi-orbit of type 2 and $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = 4$. If $v_X = 1$, $v_Y = 1$ and $p_X = -\cos(\pi/m)$ with $m \neq \infty$, then, by Equation (3-5), $\langle \tilde{e}_X, \tilde{e}_Y \rangle = -\cos(\pi/m)$. In this case (Y, X) is a bi-orbit of type 1 and $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = m$. Finally, if $v_X = v_Y = 1$ and $p_X \leq -1$, then, by Equation (3-5), $\langle \tilde{e}_X, \tilde{e}_Y \rangle = p_X \leq -1$. In this case (Y, X) is a bi-orbit of type 1 and $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = \infty$.

Case 2: X is of type II_m and Y is of type I . Applying Equations (3-2), (3-3), and (3-4) we get

$$(3-6) \quad \langle \tilde{e}_X, \tilde{e}_Y \rangle = \frac{p_X \sqrt{v_Y}}{\sqrt{v_X(1 - \cos(\pi/m))}}.$$

Applying Equation (3-1) to this equality, we get

$$(3-7) \quad \langle \tilde{e}_X, \tilde{e}_Y \rangle \leq -\frac{\sqrt{v_X v_Y}}{2\sqrt{(1 - \cos(\pi/m))}}.$$

If $m \geq 5$, then $\sqrt{1 - \cos(\pi/m)} < \frac{1}{2}$, hence, by Equation (3-7), $\langle \tilde{e}_X, \tilde{e}_Y \rangle \leq -\sqrt{v_X v_Y} \leq -1$. So, we can assume that $m \in \{3, 4\}$. Then we have $\sqrt{1 - \cos(\pi/m)} \leq \frac{1}{\sqrt{2}}$ and, by Equation (3-7), $\langle \tilde{e}_X, \tilde{e}_Y \rangle \leq -\frac{\sqrt{v_X v_Y}}{\sqrt{2}}$. It follows that, if either $v_X \geq 2$, or $v_Y \geq 2$, then

$\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -1$. If $v_X = 1$, $v_Y = 1$ and $p_X \leq -\cos(\pi/4) = -\frac{1}{\sqrt{2}}$, then, by Equation (3-6), $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -1$. If $v_X = 1$, $v_Y = 1$, $p_X = -\cos(\pi/3) = -\frac{1}{2}$ and $m = 4$, then, by Equation (3-6), $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = -\frac{\sqrt{2+\sqrt{2}}}{2} = -\cos(\pi/8)$. In this case (Y, X) is a bi-orbit of type 4 and $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = 8$. If $v_X = 1$, $v_Y = 1$, $p_X = -\cos(\pi/3) = -\frac{1}{2}$ and $m = 3$, then, by Equation (3-6), $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = -\frac{1}{\sqrt{2}} = -\cos(\pi/4)$. In this case (Y, X) is a bi-orbit of type 3 and $\tilde{m}_{Y,X} = \tilde{m}_{X,Y} = 4$.

Case 3: X is of type II_m and Y is of type $II_{m'}$. Applying Equations (3-2), (3-3) and (3-4) we get

$$\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle = \frac{p_X \sqrt{v_Y}}{\sqrt{v_X(1 - \cos(\pi/m))(1 - \cos(\pi/m'))}}.$$

Applying Equation (3-1) to this equality we get

$$\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -\frac{\sqrt{v_X v_Y}}{2\sqrt{(1 - \cos(\pi/m))(1 - \cos(\pi/m'))}}.$$

Since $\sqrt{(1 - \cos(\pi/m))} \leq \frac{1}{\sqrt{2}}$ and $\sqrt{(1 - \cos(\pi/m'))} \leq \frac{1}{\sqrt{2}}$, it follows that $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle \leq -\sqrt{v_X v_Y} \leq -1$. \square

Lemma 3.4 *Let $X \in \mathcal{S}$, and let $x \in V^G$. Then $w_X(x) = x - 2\langle x, \tilde{\epsilon}_X \rangle \tilde{\epsilon}_X$.*

Proof Let Γ' be a Coxeter graph, and let (W', S') be its associated Coxeter system, such that W' is finite. Let w'_0 be the longest element of W' , and let $(V', \langle \cdot, \cdot \rangle', \Pi')$ be the canonical root basis of Γ' . Then, by Bourbaki [1], $w'_0(\Pi') = -\Pi'$.

Let $X \in \mathcal{S}$. Recall that $\Pi_X = \{\epsilon_s \mid s \in X\}$. By Lemma 2.1 and the above, we have $w_X(\Pi_X) = -\Pi_X$, hence $w_X(a_X) = -a_X$, therefore $w_X(\tilde{\epsilon}_X) = -\tilde{\epsilon}_X$.

Recall that V_X denotes the linear subspace of V spanned by Π_X . For all $x \in V$ and all $u \in W_X$ there exists $y \in V_X$ such that $u(x) = x + y$. This is true by definition for all $s \in X$, hence it is true for all $u \in W_X$. Let $x \in V^G$. Let $y \in V_X$ be such that $w_X(x) = x + y$. Let $y = \sum_{s \in X} \lambda_s \epsilon_s$ be the expression of y in the basis Π_X . For $g \in G$ we have

$$\begin{aligned} x + \sum_{s \in X} \lambda_s \epsilon_s &= w_X(x) = g(w_X)(g(x)) = g(w_X(x)) \\ &= g(x) + \sum_{s \in X} \lambda_s g(\epsilon_s) = x + \sum_{s \in X} \lambda_s \epsilon_{g(s)}, \end{aligned}$$

hence $\lambda_s = \lambda_{g^{-1}(s)}$ for all $s \in X$. Since G acts transitively on X , it follows that $\lambda_s = \lambda_t$ for all $s, t \in X$. So, there exists $\lambda \in \mathbb{R}$ such that $w_X(x) = x + \lambda a_X = x + \lambda \|a_X\| \tilde{\epsilon}_X$.

We have

$$\langle x, \tilde{\epsilon}_X \rangle = \langle w_X(x), w_X(\tilde{\epsilon}_X) \rangle = \langle x + \lambda \|a_X\| \tilde{\epsilon}_X, -\tilde{\epsilon}_X \rangle = -\langle x, \tilde{\epsilon}_X \rangle - \lambda \|a_X\|,$$

hence $\lambda \|a_X\| = -2\langle x, \tilde{\epsilon}_X \rangle$. So, $w_X(x) = x - 2\langle x, \tilde{\epsilon}_X \rangle \tilde{\epsilon}_X$. \square

Proof of Theorem 2.2 We have $\langle \tilde{\epsilon}_X, \tilde{\epsilon}_X \rangle = 1$ for all $X \in \mathcal{S}$ by definition. We have

$$\begin{aligned} \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle &= -\cos(\pi/\tilde{m}_{X,Y}) & \text{if } \tilde{m}_{X,Y} \neq \infty, \\ \langle \tilde{\epsilon}_X, \tilde{\epsilon}_Y \rangle &\in (-\infty, -1] & \text{if } \tilde{m}_{X,Y} = \infty, \end{aligned}$$

by Lemma 3.3. Let $\chi \in V^*$ be such that $\chi(\epsilon_s) > 0$ for all $s \in \mathcal{S}$. Let $\tilde{\chi} : V^G \rightarrow \mathbb{R}$ be the restriction of χ to V^G . Then, for $X \in \mathcal{S}$, $\tilde{\chi}(\tilde{\epsilon}_X) = \frac{1}{\|a_X\|} \sum_{s \in X} \chi(\epsilon_s) > 0$. So, $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi}^G)$ is a root basis of $\tilde{\Gamma}$.

Let $(\tilde{W}, \tilde{\mathcal{S}})$ be a Coxeter system of $\tilde{\Gamma}$, where $\tilde{\mathcal{S}} = \{\tilde{s}_X \mid X \in \mathcal{S}\}$ is a set in one-to-one correspondence with \mathcal{S} . By Lemma 3.1, the map $\tilde{\mathcal{S}} \rightarrow \mathcal{S}_W$, $\tilde{s}_X \mapsto w_X$, induces a surjective homomorphism $\gamma : \tilde{W} \rightarrow W^G$. By Lemma 3.4, the composition $f^G \circ \gamma : \tilde{W} \rightarrow \text{GL}(V^G)$ is the rooted representation associated with $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$. By Theorem 1.1, it follows that $f^G \circ \gamma$ is injective, hence γ is an isomorphism. So, (W, \mathcal{S}_W) is a Coxeter system of $\tilde{\Gamma}$, and $f^G : W^G \rightarrow \text{GL}(V^G)$ is the rooted representation associated with $(V^G, \langle \cdot, \cdot \rangle, \tilde{\Pi})$. \square

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